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Goal: Understand why the Fourier Transform gives an isomorphism between $\mathbb{C}[G]$ and $End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_k})$ where $\{\rho_i\}$ are the irreducible representations of G.

Assumed Knowledge: We assume the reader is familiar with beginning Representation Theory. The following theorem we will not prove, but we will use throughout the etude:

Theorem 0.1. *Let G be a finite group. Let $\rho_1, ..., \rho_k$ be the irreducible representations of G and let $\chi_1, ..., \chi_k$ be their associated characters. Let $\mathcal{C}(G) = \{f : G \to \mathbb{C}$ such that f is a class function, i.e. $\forall g, h \in G, f(ghg^{-1}) = f(h)\}$. Define the following inner product on $\mathcal{C}(G)$:*

$$\text{Let } \phi, \psi \in \mathcal{C}(G), \text{ then } \langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g)$$

Then our theorem is that $\{\chi_1, ..., \chi_k\}$ form an orthonormal basis for $\mathcal{C}(G)$ with respect to this inner product.

Remark 0.2. From this one can show that if V is a representation of G, $V = \oplus V_i^{\oplus n_i} \iff \chi_V = \sum n_i \chi_{V_i}$

1. Introduction to Fourier Transformation at a Representation

Definition 1.1. Let G be a finite group, let $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$, and let $f \in \mathbb{C}[G]$ be any function. The fourier transform of f at the representation $\rho : G \to GL(V)$ is the matrix

$$\widehat{f}(\rho) = \sum_{g \in G} f(g)\rho(g) \in \text{End(V)}$$

.

Note: Unlike $\rho(g)$, the fourier transform of $f = \sum_{g \in G} f(g)\rho(g)$ may not be invertible nor G-linear.

So f gives us a way of associating to every representation of G ρ some linear transformation in End(V_ρ). This association is called the Fourier Transform of f and symbolically is written as $\widehat{f}$

Example (from Diaconis) A group of people ranked where they would prefer to live given the choice between a city, the suburbs, or the country. A ranking of these three options can be seen as an element of S_3. The data is below:

σ	City	Suburbs	Country	Number
id	1	2	3	242
(23)	1	3	2	28
(12)	2	1	3	170
(132)	3	1	2	628
(123)	2	3	1	12
(13)	3	2	1	359

So $f : G \to \mathbb{C}$ is $f(id) = 242$, $f((23)) = 28$, and so on.

Exercise 1.2. Using the representation matrices for S_3 given below write the Fourier Transform of f for each irreducible representation.

σ	id	(12)	(23)	(13)	(123)	(132)
$\rho_{triv}(\sigma)$	1	1	1	1	1	1
$\rho_{sgn}(\sigma)$	1	-1	-1	-1	1	1
$\rho_{stan}(\sigma)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$

Answer: You should get that $\widehat{f}(\rho_{triv}) = 1439$, $\widehat{f}(\rho_{sgn}) = 325$, and $\widehat{f}(\rho_{stan}) = \begin{bmatrix} -54.5 & 285\sqrt{3}/2 \\ -947\sqrt{3}/2 & -101.5 \end{bmatrix}$

2. Fourier Transformation as a Ring Homomorphism

Given a function f on a group G, instead of considering the fourier transform of f at *every* representation of G, let us just consider its transform on the irreducible representations of G. Let $\rho_1, ..., \rho_k$ be the irreducible representations of our finite group G and define

$$FT : \mathbb{C}[G] \to End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_k})$$

where

$$f \mapsto (\widehat{f}(\rho_1), ..., \widehat{f}(\rho_k))$$

Question 2.1. What kind of map is FT?

Exercise 2.2. First check to see that FT is a group homomorphism. So show that $FT(f_1 + f_2) = FT(f_1) + FT(f_2)$.

Next check to see that FT is a ring homomorphism. For this we need to understand both $\mathbb{C}[G]$ and $End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_k})$ as rings. That is we need to understand how elements multiply. We will multiply elements in $\mathbb{C}[G]$ as follows:

Definition 2.3. Let $f_1, f_2 \in \mathbb{C}[G]$ and define the convolution of f_1 and f_2 to be $f_1 * f_2 : G \to \mathbb{C}$ where $f_1 * f_2(g) = \sum\limits_{h \in G} f_1(gh^{-1}) f_2(h)$

Exercise 2.4. Show that for any representation ρ, $\widehat{f_1 * f_2}(\rho) = \widehat{f_1}(\rho)\widehat{f_2}(\rho)$

Exercise 2.5. Use the previous exercise to show that $FT(f_1 * f_2) = FT(f_1)FT(f_2)$. Note that if the following k-tuble of matrices $(A_1, ...A_k)$, $(B_1, ..., B_k)$ are in $End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_k})$ then $(A_1, ...A_k)(B_1, ..., B_k) = (A_1B_1, ..., A_kB_k)$.

Conclude from the exercises above that FT is a ring homomorphism.

3. Inverse Fourier Transformation

In this section we aim to show that FT is a ring isomorphism. We will show this by explicitly constructing FT^{-1}. This means that for any k-tuple of matrices in $End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_k})$ there is a unique function defined on G that will generate these transformations.

First we will look at the **regular representation**, $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$. We can think of this as the $\mathbb{C}$-span of the basis $\{e_{g_1}, ..., e_{g_k}\}$ where $e_{g_i} : G \to \mathbb{C}$ and $e_{g_i}(g_j) = \delta_{ij}$. So $f = \sum\limits_{i=1}^{k} f(g_i)e_{g_i}$
G acts naturally on this vector space: For $h \in G$, $h \cdot e_g := e_{hg}$. The regular representation captures this action in matrices. We define $\rho_{Reg} : G \to GL(V_{Reg})$ where $\rho_{Reg}(h)$ is the matrix that takes each basis vector e_g to e_{hg}. Notice then that if $hg_i = g_i$ we will have a 1 in the i^{th} row and i^{th} column of the matrix $\rho(h)$. Since G is a group, this is only the case when $h = id$. Thus we have the following:

$$\chi_{Reg}(h) = \begin{cases} 0, & \text{if } h \neq id \\ 1, & \text{if } h = id \end{cases}$$

Now we will use this to see that any irreducible representation of G appears as a subrepresentation of the regular representation.

Exercise 3.1. Let V_ρ be an irreducible representation of G. Show that $\langle \chi_{V_i}, \chi_{V_{Reg}} \rangle = \dim V_i$

From Remark 0.2 it therefore follows that any irreducible representation V_ρ of G appears as a subrepresentation of the regular representation $\dim V_\rho$ times. Thus $V_{Reg} = \oplus V_i^{\dim V_i}$. Then notice that the matrix ρ_{Reg} has a block diagonal form, where the blocks are the irreducible representations ρ_{V_i}. Each ρ_{V_i} block appears $\dim V_i$ times.

Exercise 3.2. From the above discussion, express $\text{Tr}(\widehat{f}(\rho_{Reg}))$ as a sum of $\text{Tr}(\widehat{f}(\rho_{V_i}))$ where the ρ_{V_i}s are the irreducible representations.

Answer: $\text{Tr}(\widehat{f}(\rho_{Reg}))=\text{Tr}(\sum_{g\in G} f(g)\rho_{Reg}(g)) = \sum_{g\in G} Tr(f(g)\rho_{Reg}(g)) = \sum_{g\in G}\sum_{i=1}^{k} Tr(f(g)dimV_i\rho_{V_i}(g)) = \sum_{i=1}^{k} dimV_i Tr(\sum_{g\in G} f(g)\rho_{V_i}(g)) = \sum_{i=1}^{k} dimV_i Tr(\widehat{f}(\rho_{V_i}))$ □

Exercise 3.3. Let $f \in \mathbb{C}[G]$, so $\widehat{f}(\rho_{Reg}) = \sum_{g\in G} f(g)\rho_{Reg}(g)$. Show that for a fixed $h \in G$,

$$\text{Tr}(\widehat{f}(\rho_{Reg})(\rho_{Reg}(h)^{-1})) = |G|f(h)$$

Conclude that $f(h) = \frac{1}{|G|}Tr(\widehat{f}(\rho_{Reg})\rho_{Reg}(h)^{-1})$

Exercise 3.4. Combine exercise 3.2 and 3.3 to show that for some $h \in G$, $f(h) = \frac{1}{|G|}\sum_{i=1}^{k} \text{dim}V_i Tr(\widehat{f}(\rho_{V_i})\rho_{V_i}(h)^{-1})$

Now we are able to construct the Fourier Inversion Theorem. Recall

$$FT^{-1} : End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_k}) \to \mathbb{C}[G]$$

From exercise 3.4 we see that for each $h \in G$ we can reconstruct $f(h)$ via:

$$FT^{-1}(A_1, ..., A_k) = \frac{1}{|G|}\sum_{i=1}^{k} \text{dim}V_i Tr(A_i\rho_{V_i}(h)^{-1})$$

4. Fourier Transform on Finite Cyclic Groups

Now let's consider finite cyclic groups. Let $G = \mathbb{Z}/d\mathbb{Z}$
Note: Consider a representation of G, $\rho : G \to GL(V)$. We often write that since ρ is a homomorphism, $\rho(gh) = \rho(g)\rho(h)$. But this omits the symbol for the group operation. In the case of $G = \mathbb{Z}/d\mathbb{Z}$ we have that $\rho(g + h) = \rho(g)\rho(h)$. Be aware of this throughout the following example.

First we need to show that an abelian group can only have one-dimensional irreducible representations. Recall the following:

Lemma 4.1. ***(Shur's)*** *If V and W are irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism, then*

1) Either ϕ is an isomorphism, or $\phi = 0$
2) If V=W, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$, I the identity.

Exercise 4.2. Fix some $g \in G$. Show that $\rho(g) : V \to V$ is a G-module homomorphism for every G-representation ρ (really just show that it is G-linear) if and only if $g \in Z(G) = \{h \in G | hk = kh \forall k \in G\}$

Answer: Fix some $g \in G$ and let ρ be the regular representation. Suppose that $\rho(g) : V \to V$ is G-linear. Then $\forall h \in G$, $\rho(g)(\rho(h)(v)) = \rho(h)(\rho(g)(v))$ Since ρ is a homomorphism, $\rho(g)\rho(h) = \rho(gh)$. Thus

$$\rho(gh(v)) = \rho(hg)(v)$$
$$\rho(gh)\rho(hg)^{-1} = I$$
$$\rho(ghg^{-1}h^{-1}) = I$$

Since ρ is the regular representation, from our discussion at the beginning of section three, we have that $ghg^{-1}h^{-1} = id$. Thus $g \in Z(G)$
Now suppose that $g \in Z(G)$, then for any representation we have that $\rho(gh)(v) = \rho(hg)(v)$ and since ρ is a homomorphism we get that $\rho(g)(\rho(h)(v)) = \rho(h)(\rho(g)(v))$

Question 4.3. If G is abelian then what is $Z(G)$? What does this mean about $\rho(g) : V \to V$ for any $g \in G$?

Question 4.4. Suppose $\rho : G \to GL(V)$ is an irreducible representation. Moreover, suppose G is abelian. Using what you know about $\rho(g) : V \to V$ and part (2) of Schur's lemma then what can you conclude about the dimension of V?

Answer: Since G is abelian, $\forall g \in G$, $\rho(g) : V \to V$ is a G-module homomorphism. Then by (2) of Schur's lemma $\rho(g)$ is some scalar multiple of the identity matrix for any g. Thus any subspace of V is G-invariant. Since V is irreducible we conclude that $dimV = 1$.

Let's get back to our example of $G = \mathbb{Z}/d\mathbb{Z}$. Its irreducible represenations are one dimensional, i.e. $\forall k \in \mathbb{Z}/d\mathbb{Z}$, $\rho(k) \in \mathbb{C}$. Notice that $dk = id$ so $\rho(k)^d = I$ and so $\rho(k)$ is a d^{th} root of unity. Thus $\rho(k) \in \mathbb{C}^*$

Exercise 4.5. Show that $\rho_j : G \to \mathbb{C}$, where $\rho_j(k) = e^{\frac{2\pi ijk}{d}}$ is an irreducible representation for $j = 1...d$

Do we have all of the irreducible representations? Recall that the characters of the irreducible representations for a basis for $\mathcal{C}(G)$, the class functions on G. Since G is abelian, how many conjugacy classes are there? Thus how many irreducible representations are there?

Consider our Fourier Transform map,

$$FT : \mathbb{C}[G] \to End(V_{\rho_1}) \oplus ... \oplus End(V_{\rho_d})$$

where

$$f \to (\widehat{(f)}(\rho_1), ..., \widehat{f}(\rho_d))$$

If $f \in \mathbb{C}[\mathbb{Z}/d\mathbb{Z}]$, then $f(k + nd) = f(k)$, $\forall n \in \mathbb{N}$, so $\mathbb{C}[\mathbb{Z}/d\mathbb{Z}]$ contains the functions with period d. As for the domain we have that an irreducible representation is of the form $\rho_j : G \to \mathbb{C}^*$. Thus $End(V_{\rho_1}) = End(\mathbb{C}^*) = \mathbb{C}^*$. Thus FT takes periodic functions to a d-tuple with elements on the unit circle. And with the inverse transform we know that a sequence of d values on the unit circle corresponds uniquely to a periodic function (with period d).
$FT : \mathbb{C}[\mathbb{Z}/d\mathbb{Z}] \to (\mathbb{C}^*)^{\oplus d}$

Exercise 4.6. Determine the fourier transform of f at ρ_k.

Answer: The fourier transform of f at ρ_k is given by:

$$\widehat{f}(\rho_j) = \sum_{m=1}^{d} f(m) e^{\frac{2\pi imk}{d}}$$

Exercise 4.7. Give a formula for the inverse transform FT^{-1}. If $(a_1, ..., a_d) \in (\mathbb{C}^*)^{\oplus d}$, then what is $f \in \mathbb{C}[\mathbb{Z}/d\mathbb{Z}]$

Answer: For $k \in \mathbb{Z}/d\mathbb{Z}$ we have that $f(k) = \frac{1}{d} \sum\limits_{j=1}^{d} a_j e^{-\frac{2\pi ijk}{d}}$